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A VARIATIONAL PRINCIPLE FOR THE LAPLACE'S OPERATOR WITH APPLICATION IN THE TORSION OF COMPOSITE RODS†

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Abstract—The Ritz method with relaxed coordinate functions is used here to establish a minimizing sequence for an extended functional. The integrand of the functional involves discontinuous functions, which implies that the gradient of the related extremal curve or surface also admits discontinuities. It is discussed a systematic way to form an extended functional allowing the minimizing sequence to approach the exact solution displaying the discontinuities in the gradient of the extremal surface. The extra terms added to the classical functional are shown to be related to the Erdman–Weierstrass corner conditions. Finally the method is applied to the torsion of a composite rod, and the results compared with a solution obtained through the finite elements method.

1. INTRODUCTION

Variational principles have been widely used in the analysis of static and dynamical problems in solid mechanics in recent times[1].

The effectivness of the direct methods for the minimization of a functional such as the Ritz, Galerkin, finite elements methods and others, together with an increasing elaboration in the computational techniques brought a revival of the extremal principles. It is well known that almost all of the finite element techniques can be derived from a variational principle in a very elegant way. This tendency is to be found in the recent literature[2–4].

The scope of the direct methods in the calculus of variations is to find a series representation for the extremal of a given functional $J[\mathbf{u}]$:

$$J[\mathbf{u}] = \int_D F\left(u_i, \frac{\partial u_i}{\partial x_j}, x_j\right) \mathrm{d}\Omega \qquad \begin{matrix} i = 1, n \\ j = 1, m \end{matrix} \tag{1}$$

$$\mathbf{u}|_{\partial D} = \mathbf{u}_0 \tag{2}$$

where $\mathbf{u}_0$ is given on the boundary. Now, if the functional (1) with the auxiliary condition (2) admits a minimum $\mathbf{u}^*(\mathbf{x})$, $J[\mathbf{u}^*] \leqslant J[\mathbf{u}]$ and if we require further, that the integrand in (1) is

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continuous in all its variables, then the extremal $\mathbf{u}^*(\mathbf{x})$ can be approximated by a minimizing sequence of the form:

$$\mathbf{u}_n(\mathbf{x}) = \sum_{i=1}^{n} \mathbf{A}_i \boldsymbol{\phi}_i(\mathbf{x}) \tag{3}$$

where the $\boldsymbol{\phi}_i(\mathbf{x})$ are the coordinate vector functions of a n-dimensional function space. We assume further that this is a normed space. If the admissible function space M: $\{\boldsymbol{\phi}_i(\mathbf{x})\}$ is complete, it can be shown[5] that the sequence (3) converges to $\mathbf{u}^*(\mathbf{x})$ with respect to the norm of M.

This is a central point in the variational technique. In general we can only claim that the sequence (3) will approach the solution $\mathbf{u}^*(\mathbf{x})$, in a sense that the measure of the closeness is a certain pre-established norm. Except for particular cases, the pointwise convergence is ruled out, and we can expect that the sequence (3) approaches $\mathbf{u}^*(\mathbf{x})$ in the sense of the mean convergence or energy convergence[5].

Suppose now, that besides $\mathbf{u}^*(\mathbf{x})$ we want also a representation for $\partial \mathbf{u}/\partial x_i$. Then the situation is even more critical. We are not allowed to differentiate the sequence (3) term by term, with respect to x_i, make the limit as $n \to \infty$, and claim that this sequence converges to $\partial \mathbf{u}/\partial x_i$, without proving it. In general this is not true.

Let us take for instance a very simple, but illustrative example. Consider a rod of uniform cross section with area equals unit, composed by two different materials as depicted in Fig. 1.

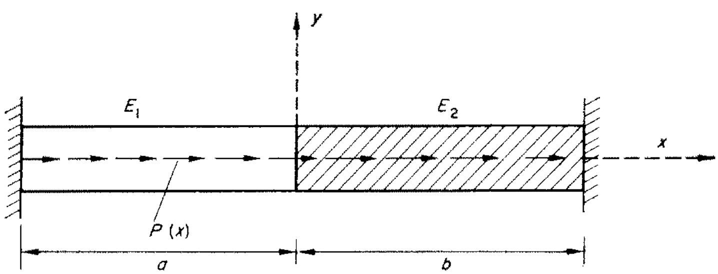

Fig. 1. Non-homogeneous rod under an axial loading.

Assume that it is loaded by a longitudinal distributed force $p(x)$ and fixed at both ends. If $u(x)$ represents the longitudinal displacement of a cross section, the actual solution minimizes the functional:

$$J[u] = \frac{1}{2}\int_{-a}^{b} \left[E(x)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - 2p(x)u(x)\right]\mathrm{d}x \qquad -a < x < b \tag{4}$$

where the integral is to be understood as the union of the integrals over the intervals $[-a, 0)$ and $(0, b]$.

$$u(-a) = 0 \tag{5a}$$

$$u(b) = 0 \tag{5b}$$

$$E(x) = \begin{cases} E_1 & -a \leqslant x < 0 \\ E_2 & 0 < x \leqslant b. \end{cases} \tag{6}$$

The minimizing sequence

$$u_n(x) = \sum_{i=0}^{n} a_i \sin \frac{i\pi x}{a+b} \qquad -a < x < b \tag{7}$$

fulfils the boundary conditions (5a) and (5b), and approaches the solution $u^*(x)$ in the mean. Now, the strain $\varepsilon(x) = \mathrm{d}u/\mathrm{d}x$ cannot be obtained by differentiating the sequence (7). Indeed, it is clear that the strain at the origin to the left is different from the strain at the origin to the right, that is:

$$\varepsilon(0^-) \neq \varepsilon(0^+) \tag{8}$$

since the Young moduli are distinct for both materials and the continuity condition of the stress field at the interface requires that:

$$E_1 \varepsilon(0^-) = E_2 \varepsilon(0^+) \tag{9a}$$

hence

$$\frac{\varepsilon(0^-)}{\varepsilon(0^+)} = \frac{E_2}{E_1} \neq 1. \tag{9b}$$

On the other hand, the space of admissible functions is formed by the set of infinitely many differentiable continuous functions $\{\sin i\pi x/a + b\}$. Therefore:

$$\frac{\mathrm{d}u_n(0^-)}{\mathrm{d}x} = \frac{\mathrm{d}u_n(0^+)}{\mathrm{d}x} \tag{10}$$

for every n, and (10) is also true as $n \to \infty$. That is the derivative of the sequence, does not converge to the derivative of the solution, at least at $x = 0 + \delta$ and $x = 0 - \delta$, where δ is any arbitrary small positive number. This fact has already been pointed out[6] in connection with wave propagation in composite material.

Although it is true that in many cases, an approximation in the sense of the convergence in the mean or in the energy is sufficient, nevertheless it would be useful to have a minimizing sequence that could exhibit the actual discontinuities in the derivatives of the extremal $u(x)$. It has been shown[6, 7] that adding some complementary terms to the energy functional it is possible to have a series representation that exhibit these discontinuities. A systematic way of establishing an extended functional and a field of admissible functions in order to achieve a representation that can display jumps in the derivatives, is the purpose of this paper. Prager[8] and Leipholz[9] have also examined the problem of discontinuities of the extremal and relaxing the boundary conditions, but from a quite different point of view.

2. BROKEN EXTREMAL FOR A PARTICULAR CLASS OF FUNCTIONALS

In this paper we will restrict ourselves to boundary value problems of the kind:

$$\nabla(g(x, y)\,\nabla u) + \lambda u = f(x, y) \qquad x, y \in D \tag{11}$$

$$u(x, y)|_{\partial D} = 0 \tag{12a}$$

or

$$\left.\frac{\partial u}{\partial n}\right|_{\partial D} = 0 \tag{12b}$$

where ∇ is the gradient operator, $g(x, y)$ and $f(x, y)$ are sectionally continuous functions in a closed domain D.

It can be shown[5] that if $u^*(x, y)$ is the solution to the partial differential equation (11) together with the boundary conditions (12a) or (12b) it also minimizes the functional†:

$$J[u] = \frac{1}{2}\iint_D [g(\nabla u)^2 - \lambda u^2 + 2uf]\,\mathrm{d}x\,\mathrm{d}y \tag{13}$$

with one of the boundary conditions (12a) or (12b). In this case, however, the gradient of $u^*(x, y)$ is discontinuous on the lines where $g(x, y)$ is discontinuous. Indeed, let $g(x, y)$ be discontinuous on the curve C_{12} (Fig. 2) which divides the domain D into two subdomains

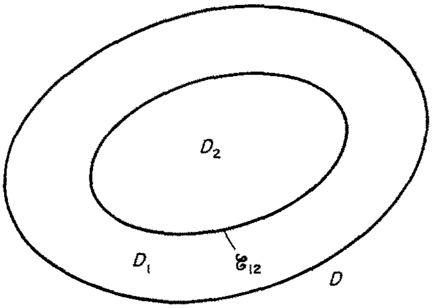

Fig. 2.

D_1 and D_2, such that $D = D_1 \cup D_2$. Call P: (x^*, y^*) a generic point on C_{12} and let us agree to designate by P_1 and P_2 respectively, points of D_1 and D_2 whose distances from P are arbitrary small. Then $g(x, y)$ can be defined as follows:

$$g(x, y) = \begin{cases} g_1(x, y) & (x, y) \in D \\ g_2(x, y) & (x, y) \in D \end{cases}$$

and $\lim_{P_1 \to P} g_1(P_1) \neq \lim_{P_2 \to P} g_2(P_2)$.

The functional (13) can now be rewritten:

$$J[u] = \frac{1}{2}\iint_{D_1} (g_1(\nabla u)^2 - \lambda u^2 + 2uf)\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\iint_{D_2} (g_2(\nabla u)^2 - \lambda u^2 + 2uf)\,\mathrm{d}x\,\mathrm{d}y. \tag{14}$$

Taking the first variation we obtain:

$$\delta J[u] = \iint_{D_1} g_1 \nabla u \,.\, \nabla(\delta u) - \lambda u\,\delta u + f\delta u)\,\mathrm{d}x\,\mathrm{d}y + \iint_{D_2} (g_2 \nabla \mathbf{u} \,.\, \nabla(\delta u) - \lambda u\,\delta u + f\delta u)\,\mathrm{d}x\,\mathrm{d}y$$

and using Green's theorem:

$$\delta J[\mathbf{u}] = \iint_{D_1} [-\nabla(g_1 \nabla u) - \lambda u + f]\,\delta u\,\mathrm{d}x\,\mathrm{d}y + \iint_{D_2} [-\nabla(g_2\,\nabla u) - \lambda u + f]\,\delta u\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \int_{\partial D} g_1 \frac{\partial u}{\partial n}\,\delta u\,\mathrm{d}s - \int_{C_{12}} \left[g_1 \frac{\partial u}{\partial n}\,\delta u\right]_{P_1 \to P} \mathrm{d}s + \int_{C_{12}} \left[g_2 \frac{\partial u}{\partial n}\,\delta u\right]_{P_2 \to P} \mathrm{d}s. \tag{15}$$

† The integral (13) is to be understood as the union of the integrals over the sub-domains of D where $g(x, y)$ is continuous.

But in view of the boundary conditions (12a) or (12b) either $\delta u|_{\partial D} = 0$ or $\left.\frac{\partial u}{\partial n}\right|_{\partial D} = 0$, therefore the above expression reduces to:

$$\delta J[u] = \iint_D [-\nabla(g\nabla u) - \lambda u + f]\,\delta u\,\mathrm{d}x\,\mathrm{d}y + \int_{C_{12}} \left\{ \left[g_2 \frac{\partial u}{\partial n} \right]_{P_2 \to P} - \left[g_1 \frac{\partial u}{\partial n} \right]_{P_1 \to P} \right\} \delta u\,\mathrm{d}s. \tag{16}$$

We note that the field of admissible functions is continuous and then $\lim_{P_1 \to P} \delta u(P_1) = \lim_{P_2 \to P} \delta u(P_2)$, which allows the rearrangement of the two last integrals in (15) into the last integral in (16).

A necessary condition for the minimum is $\delta J[u] = 0$, and since δu is arbitrary the fundamental lemma of the calculus of variations leads to:

$$\nabla(g(x, y)\nabla u) + \lambda u = f(x, y) \tag{17}$$

$$\left[g_2 \frac{\partial u}{\partial n} \right]_{P_2 \to P} - \left[g_1 \frac{\partial u}{\partial n} \right]_{P_1 \to P} = 0. \tag{18}$$

The first equation is clearly the differential equation (11) and the second equation gives the jump relation for the directional derivative of u along C_{12}. This relation is equivalent to the first Erdman–Weierstrass corner condition[10] for broken extremals.

If the functional (13) *defined in a domain* D *admits a minimum, and* g(x, y) *is discontinuous on a curve* C_{12} *which divides* D *into two disconnected subdomains* D_1 and D_2, *and is a continuous differentiable function elsewhere in* D, *then the gradient of the extremal surface* u*(x, y) *has a jump discontinuity on* C_{12} *such that*:

$$\left[g_2 \frac{\partial u}{\partial n} \right]_{P_2 \to P} - \left[g_1 \frac{\partial u}{\partial n} \right]_{P_1 \to P} = 0$$

where $\partial u/\partial n$ means the directional derivative normal to C_{12}.

It is clear then that all competing functions have to satisfy the relation (18). It is easy to realize now, that it would be very difficult to select a minimizing sequence that fulfills this requirement. It is however possible to overcome this difficulty by modifying conveniently the functional (13).

3. THE EXTENDED FUNCTIONAL. RELAXING THE CORNER CONDITION

It has been shown[7], that if certain terms are added to the classical energy functional related to the wave propagation in solids, the competing functions do not have to satisfy the Erdman–Weierstrass corner condition.

We want to set up a similar extended functional related to equation (11).

Consider the functional

$$I[u] = \frac{1}{2} \iint_D [g(\nabla u)^2 - \lambda u^2 + 2uf]\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2} \int_{C_{12}} \left\{ \left[g_2 \frac{\partial u}{\partial n} \right]_{P_2 \to P} + \left[g_1 \frac{\partial u}{\partial n} \right]_{P_1 \to P} \right\} [u(P_2) - u(P_1)]_{P_1, P_2 \to P}\,\mathrm{d}s \tag{19}$$

$$u(x, y)|_{\partial D} = 0 \quad \text{or} \quad \left.\frac{\partial u}{\partial n}\right|_{\partial D} = 0$$

where $g(x, y)$ and C_{12} are defined as before. The class of admissible functions for this functional can be extended in order to include discontinuous functions. Let $v(x, y)$ be defined as follows:

(a) $v(x, y) = \begin{cases} v_1(x, y) & x, y \in D_1 \\ v_2(x, y) & x, y \in D_2 \end{cases}$

(b) $v_1(x, y)$ and $v_2(x, y)$ are continuous differentiable functions in D_1 and D_2 respectively

(c) $[v_1]_{P_1 \to P} \neq [v_2]_{P_2 \to P}$

$$\left[\frac{\partial v_1}{\partial n}\right]_{P_1 \to P} \neq \left[\frac{\partial v_2}{\partial n}\right]_{P_2 \to P}$$

(d) $v(x, y)|_{\partial D} = 0$ or $v(x, y)|_{\partial D} \neq 0$.

Then $v(x, y)$ belongs to the class of admissible functions. Indeed, the first variation of (19) gives:

$$\begin{aligned} \delta I = & \iint_D (g \, \nabla u \, . \, \nabla(\delta u) - \lambda u \, \delta u + f \, \delta u) \, dx \, dy \\ & - \frac{1}{2} \int_{C_{12}} \left[\left(g_2 \frac{\partial \delta u_2}{\partial n} + g_1 \frac{\partial \delta u_1}{\partial n} \right) (u(P_2) - u(P_1)) \right. \\ & \left. - \left(g_2 \frac{\partial u}{\partial n} + g_1 \frac{\partial u}{\partial n} \right) (\delta u_2 - \delta u_1) \right]_{P_{1,2} \to P} ds \end{aligned}$$

where δu is defined in the same way as v. This expression can be rewritten after using Green's theorem:

$$\begin{aligned} \delta I = & \iint_D - \nabla(g \, \nabla u) - \lambda u + f] \, \delta u \, dx \, dy \\ & - \frac{1}{2} \int_{C_{12}} \left[\left(g_2 \frac{\partial u}{\partial n} - g_1 \frac{\partial u}{\partial n} \right) (\delta u_2 + \delta u_1) \right]_{P_{1,2} \to P} ds \\ & - \frac{1}{2} \int_{C_{12}} \left[\left(g_2 \frac{\partial \delta u_2}{\partial n} + g_1 \frac{\partial \delta u_1}{\partial n} \right) (u(P_2) - u(P_1)) \right]_{P_{1,2} \to P} ds. \end{aligned} \tag{20}$$

But the admissible functions $\delta u = v(x, y)$ are free on C_{12}, that is, there are no requirements on the continuity of the function or its derivatives on C_{12}. Therefore the minimum of the functional (19) will be reached for $\delta I = 0$, that is, when expression (20) vanishes. The first integrand in (20) clearly vanishes, and since δu is arbitrary it is necessary that

$$\left[g_2 \frac{\partial u}{\partial n} - g_1 \frac{\partial u}{\partial n} \right]_{P_{1,2} \to P} = 0$$

$$[u(P_2) - u(P_1)]_{P_{1,2} \to P} = 0$$

for $\delta I = 0$.

The second relation implies that $u(x, y)$ is continuous on C_{12} and the first one is the first condition of Erdman–Weierstrass that allows for a jump discontinuity on the gradient of $u(x, y)$ on C_{12}.

The minimum of the functional (19) can then be approximated by a minimizing sequence discontinuous on C_{12}. We can select a set $\{\phi_k(x, y)\}$ of coordinate functions defined on D_1

and another set $\{\psi_k(x, y)\}$ defined on D_2, such that $[\phi_k(P_1)]_{P_1 \to P} \neq [\psi_k(P_2)]_{P_2 \to P}$ and form the minimizing sequence $u_n(x, y)$:

$$u_n = \begin{cases} \sum_{k=1}^{n} A_k \phi_k & (x, y) \in D_1 \\ \sum_{k=1}^{n} B_k \psi_k & (x, y) \in D_2 . \end{cases} \tag{21}$$

The Ritz method to evaluate the A_k's and B_k's such that (19) reaches a minimum, leads automatically to those coefficients related to the extremal surface continuous on C_{12} and with a discontinuity on the gradient on C_{12} as given by (18).

Although the numerical computation is more involved for the functional (19) as compared to the functional (13), nevertheless it allows an excellent approximation to the minimizing functions using a discontinuous sequence (21). As it will be seen in the next section the jump discontinuity in the normal derivative along C_{12} is clearly displayed by the minimizing sequence.

4. TORSION OF A NONHOMOGENEOUS ROD

Consider a rod of finite length with rectangular cross-section consisting of two different perfectly elastic materials as shown in Fig. 3. It is subjected to a torque at both ends according to Saint-Venant's theory.

Let $\Phi(x, y)$ be the well known stress function for the torsion of prismatical bars. The shear stresses and the torque in terms of $\Phi(x, y)$ are given respectively by:

$$\tau_{xz} = \frac{\partial \Phi}{\partial y} \tag{22a}$$

$$\tau_{yz} = -\frac{\partial \Phi}{\partial x} \tag{22b}$$

$$M_t = 2 \iint \Phi(x, y) \, dx \, dy. \tag{23}$$

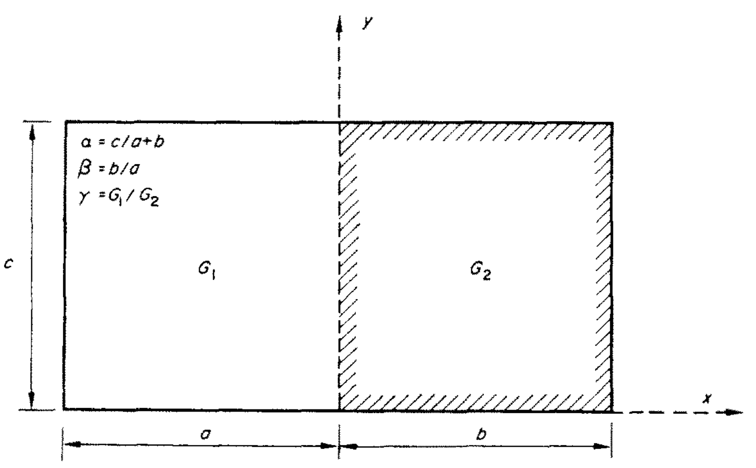

Fig. 3. Cross section of the composite rod.

If $\Phi(x, y)$ is the solution to the torsion problem it minimizes a functional similar to that in (13), with $\lambda = 0$, $1/g(x, y)$ and $-1/2 f(x, y)$ representing the shear modulus and the specific rotation of the cross section respectively. The boundary condition is $\Phi(x, y)|_{\partial D} = 0$. But in the present case the shear modulus is discontinuous on $x = 0$. Consequently, the shear stress τ_{yz} is discontinuous on $x = 0$. Then as it is clear from (22), the gradient of the stress function $\Phi(x, y)$ is also discontinuous on $x = 0$. In order to make apparent this discontinuity in the solution, let us form the extended functional according to the theory presented in the Section 3. We may write:

$$I[\Phi] = \frac{1}{2}\int_{-a}^{b}\int_{0}^{c}\left(\frac{1}{G}(\nabla\Phi)^2 - 4\theta\Phi\right)\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\int_{0}^{c}\left\{+\left[\frac{1}{G_1}\frac{\partial\Phi^{(1)}}{\partial x} + \frac{1}{G_2}\frac{\partial\Phi^{(2)}}{\partial x}\right](\Phi^{(2)} - \Phi^{(1)})\right\}\mathrm{d}y \tag{24}$$

where the specific rotation θ is constant and the shear modulus $G(x, y)$ is equal to G_1 in D_1 and G_2 in D_2, $G_1 \neq G_2$. $\Phi^{(1)}$ and $\Phi^{(2)}$ represent respectively $\lim_{x\to 0^-} \Phi(x, y)$ and $\lim_{x\to 0^+} \Phi(x, y)$. Let us use the Ritz's method to form a discontinuous minimizing sequence:

$$\frac{\Phi(x, h)}{G_2\theta a^2} \sim \begin{cases} \sum\limits_{\mu,\nu=1}^{m,n} A_{\mu\nu}^{(1)} \sin\dfrac{\mu\pi(y-c)}{c}\sin\dfrac{\nu\pi(x+a)}{a+b} & -a \leqslant x \leqslant 0 \\ & 0 \leqslant y \leqslant c \quad (25a) \\ \sum\limits_{,\nu=1}^{m,n} A_{\mu\nu}^{(2)} \sin\dfrac{\mu\pi(y-c)}{c}\sin\dfrac{\nu\pi(b-x)}{a+b} & 0 \leqslant x \leqslant b \\ & 0 \leqslant y \leqslant c. \quad (25b) \end{cases}$$

Introducing (25a) and (25b) in (24) and using the conditions for a minimum in terms of $A_{\mu\nu}^{(1)}$ and $A_{\mu\nu}^{(2)}$ we obtain:

$$C_{\mu k} A_{k\nu}^{(1)} + D_{\mu k} A_{k\nu}^{(2)} = S_{\mu\nu}(\mu, k = 1, \ldots m)$$
$$E_{\mu k} A_{k\nu}^{(1)} + F_{\mu k} A_{k\nu}^{(2)} = T_{\mu\nu}(\mu, k = 1, \ldots m)$$

for a fixed value of ν.
where:

$$C_{\mu k} = \mu k\pi^2 H_{\mu k}^{(1)} + \frac{\nu^2\pi^2}{\alpha^2} H_{\mu k}^{(2)} - \mu\pi\cos\frac{\mu\pi}{1+\beta}\sin\frac{k\pi}{1+\beta} - k\pi\sin\frac{\mu\pi}{1+\beta}\cos\frac{k\pi}{1+\beta}$$

$$D_{\mu k} = \mu\pi\cos\frac{\mu\pi}{1+\beta}\sin\frac{k\pi\beta}{1+\beta} + k\pi\gamma\sin\frac{\mu\pi}{1+\beta}\cos\frac{k\pi\beta}{1+\beta}$$

$$E_{\mu k} = \mu\pi\cos\frac{\mu\pi\beta}{1+\beta}\sin\frac{k\pi}{1+\beta} + \frac{k\pi}{\gamma}\sin\frac{\mu\pi\beta}{1+\beta}\cos\frac{k\pi}{1+\beta}$$

$$F_{\mu k} = \mu k\pi^2 L_{\mu k}^{(1)} + \frac{\nu^2\pi^2}{\alpha^2} L_{\mu k}^{(2)} - \mu\pi\cos\frac{\mu\pi\beta}{1+\beta}\sin\frac{k\pi\beta}{1+\beta} - k\pi\sin\frac{\mu\pi\beta}{1+\beta}\cos\frac{k\pi\beta}{1+\beta}$$

$$S_{\mu\nu} = \begin{cases} 0 & \text{for } \nu \text{ even} \\ -16\gamma\dfrac{(1+\beta)^2}{\mu\nu\pi^2}\left(1 - \cos\dfrac{\mu\pi}{1+\beta}\right) & \text{for } \nu \text{ odd} \end{cases}$$

$$T_{\mu\nu} = \begin{cases} 0 & \text{for } \nu \text{ even} \\ -16\dfrac{(1+\beta)^2}{\mu\nu\pi^2}\left(1 - \cos\dfrac{\mu\pi\beta}{1+\beta}\right) & \text{for } \nu \text{ odd} \end{cases}$$

and

$$H_{\mu k}^{(1),(2)} \begin{cases} \dfrac{1}{1+\beta} & \text{for } k = \mu \\ \dfrac{\sin\dfrac{\pi(k-\mu)}{1+\beta}}{\pi(k-\mu)} \pm \dfrac{\sin\dfrac{\pi(k+\mu)}{1+\beta}}{\pi(k+\mu)} & \text{for } k \neq \mu \end{cases}$$

$$L_{\mu k}^{(1),(2)} = \begin{cases} \dfrac{\beta}{1+\beta} & \text{for } k = \mu \\ \dfrac{\sin\dfrac{\pi(k-\mu)\beta}{1+\beta}}{\pi(k-\mu)} \pm \dfrac{\sin\dfrac{\pi(k+\mu)\beta}{1+\beta}}{\pi(k+\mu)} & \text{for } k \neq \mu \end{cases}$$

$$\alpha = \frac{c}{a+b}, \quad \beta = \frac{b}{a}, \quad \gamma = \frac{G_1}{G_2}.$$

In order to test the accuracy and convergence of the Ritz method, it was compared to a finite element solution.

For a non-homogeneous cross section with $\alpha = 0{\cdot}5$, $\beta = 1{\cdot}0$ and $\gamma = 10{\cdot}0$ the reduced stress-function and shear stress for $x = 0$ are shown in Figs. 4–6. The solution obtained by the Ritz method with 16 terms in the series and by a finite element technique using a simple triangular element, with a linear interpolation function for the reduced stress function[13], with 200 elements in one half of the cross section, are in very good agreement.

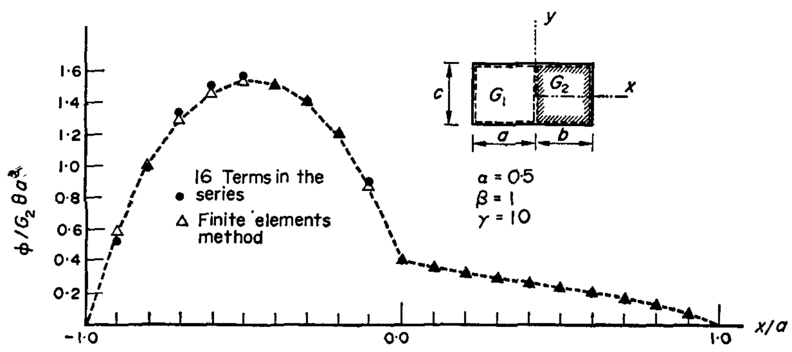

Fig. 4. Profile of the reduced stress function at $y = 0$, in the interval $-a \leqslant x \leqslant a$.

The discontinuity in the first derivative of the stress function along the x axis, is clearly exhibited in Fig. 4. A comparison between the values taken by the reduced stress function to the left and to the right of $x = 0$ (Fig. 7) shows that, for 16 terms in the series, the maximum error is less than 2 per cent. For 36 terms in the series the error is almost undetectable.

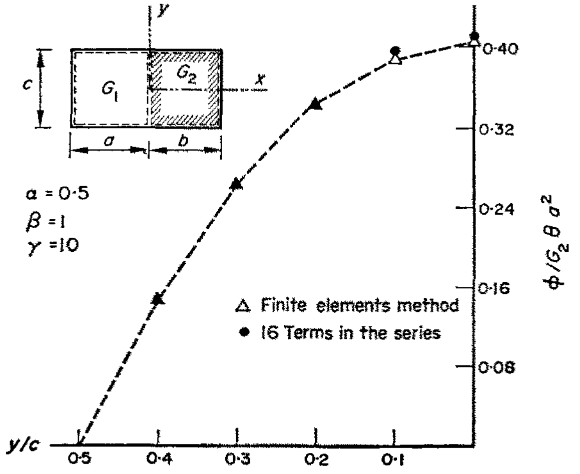

Fig. 5. Profile of the reduced stress function at $x = 0$.

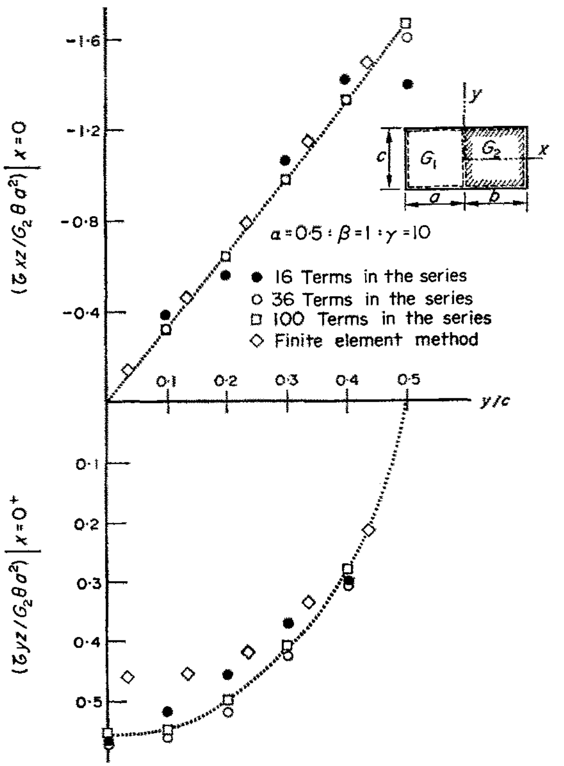

Fig. 6. Reduced shear stress at the interface.

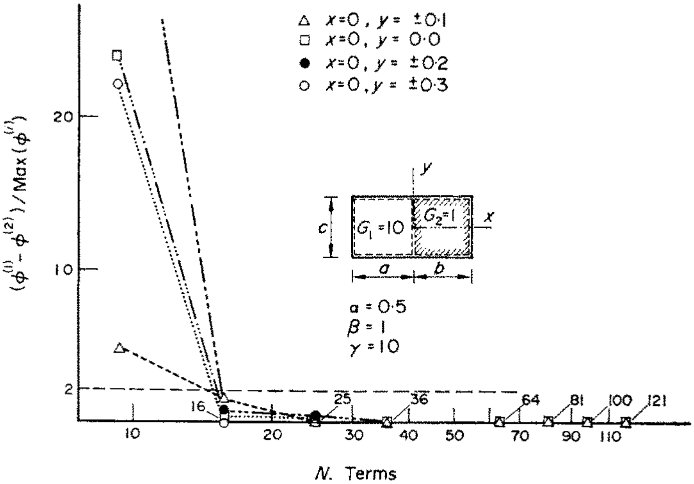

Fig. 7. Relative error on the approximation of the stress function at the interface.

The torsional rigidity for some values of the ratios γ and β was compared to a solution given by Muskhelishvili[11]. For all cases examined, the error was less then 2 per cent using 16 terms in the series (Fig. 8).

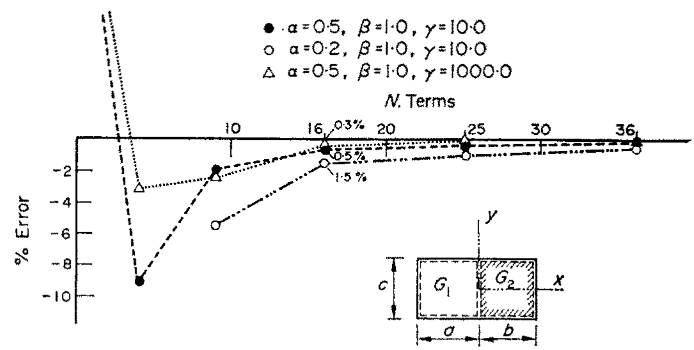

Fig. 8. Relative error of the torsional rigidity obtained with the Ritz method compared with the Muskhelishvili's solution.

5. CONCLUSIONS

The Ritz method applied to the extended functional provides an excellent approximation for two dimensional problems. The discontinuity in the first derivative of the extremal surface is also quite clearly shown in the series solution. This result confirms the efficiency of these extended functionals for two-dimensional problems.

The determination of extended functionals for other class of problems, involving derivatives of higher order for instance, can be done starting from the Erdman–Weierstrass corner conditions[12], and following the same approach discussed above.

It seems also that the use of such functionals to derive the fundamental equations of a finite element technique is promising, in those cases where it is convenient to relax the requirements of continuity of some of the variables.

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Абстракт — Применяется здесь метод Ритца с ослабленными координатными функциями, с целью определения доведенной до минимума последовательности для функционала уточнения. Интегрируемая функция функционала заключает в себе разрывные функции. Это является причиной, что градиент для зависимой экстремальной кривой или поверхности также допускает разрывы.

Исследуется систематическая путь для оформления функционала уточнения, разрешающая минимальной последовательности приблизится к строгому решению, показывающего разрывы в градиенте экстремальной поверхности.

Указано, что добавочный член, прибавленный к классическому решению, связан с угловым условием ЭрдименаУэфстрана.

Окончательно, применяется этот метод к кручению составного стержня. Сравниваются результаты с решением полученным методом конечного элемента.